Calculus 1

Midterm Exam – Solutions
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1) Prove using the ε-δ definition that $\lim_{x\to 4} \dfrac{x^2-2x-8}{x-4} = 6$.

Solution. Let $\varepsilon > 0$ be arbitrary and take $\delta = \varepsilon$. Then, $0 < |x-4| < \delta$ implies that

$$\left|\frac{x^2-2x-8}{x-4} - 6\right| = \left|\frac{(x-4)(x+2)}{x-4} - 6\right| = |(x+2)-6| = |x-4| < \delta = \varepsilon.$$

Thus, $\lim_{x\to 4} \dfrac{x^2-2x-8}{x-4} = 6$.

2) Apply l'Hospital's Rule to evaluate the following limit: $\lim_{x\to 0}[\cos(x)]^{\frac{1}{x^2}}$. Indicate which rules of differentiation are being applied.

Solution. This limit is an indeterminate form of type "1^∞" since $\cos 0 = 1$ and $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$. Notice however that moving $\cos(x)$ to the exponent by writing

$$[\cos(x)]^{\frac{1}{x^2}} = \exp\left(\frac{1}{x^2}\ln(\cos(x))\right),$$

where $\exp(x) = e^x$ is used to keep things readable, the exponent has an indeterminate form as desired, specifically "$0/0$". We note that we are allowed to do this because when x is near 0, $\cos(x)$ is near 1, so the value of $\ln(\cos(x))$ exists. Recall that if f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b).$$

[This is Theorem 8 on page 120 of the textbook.] In our case, $f(x) = e^x$ is continuous everywhere, hence, if $\lim_{x\to 0} \dfrac{\ln(\cos(x))}{x^2}$ exists, then we can compute the limit by the above method. The limit of the exponent as $x \to 0$ can by found using l'Hospital's Rule twice:

$$\lim_{x\to 0}\frac{\ln(\cos(x))}{x^2} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{\frac{-\sin(x)}{\cos(x)}}{2x} = \lim_{x\to 0}\frac{-\tan(x)}{2x} \overset{\text{l'H}}{=} \lim_{x\to 0}\frac{-\frac{1}{\cos^2(x)}}{2} = -\frac{1}{2}.$$

At the first application of l'Hospital's Rule we used the Chain Rule, the derivatives $(\ln x)' = 1/x$, $(\cos x)' = -\sin x$, and the Power Rule $(x^2)' = 2x$. The second time we used l'Hospital's Rule we employed the derivatives $(\tan x)' = \sec^2 x$, $(x)' = 1$, and the Constant Multiple Rule. The last equality follows by direct substitution. Therefore we have

$$\lim_{x\to 0}[\cos(x)]^{\frac{1}{x^2}} = \exp\left(\lim_{x\to 0}\frac{\ln(\cos(x))}{x^2}\right) = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

3) Two curves are *orthogonal* if their tangent lines are perpendicular at each point of intersection. Determine the value of the number a such that the curves $xy = 1$ and $y^3 = a^3 x$ are orthogonal.

Solution. We begin by noting that two lines are perpendicular iff the product of their slopes is -1. This can be shown in various ways. For example, if the angle of inclination of one line is θ, the other line must have $\frac{\pi}{2} - \theta$ for its angle of inclination so that these angles add up to $\frac{\pi}{2}$ radians ($90°$). The slope of the first line is $\tan(\theta)$ whereas the second line has slope $\tan(\frac{\pi}{2} - \theta) = -\cot(\theta) = -\frac{1}{\tan(\theta)}$.

Next, we will calculate the derivatives for each curve using the Generalized Power Rule. We have $y = x^{-1} \Rightarrow y' = -x^{-2}$ and $y = ax^{1/3} \Rightarrow y' = \frac{1}{3}ax^{-2/3}$. Hence the product of the slopes of tangent lines is $\frac{-1}{x^2} \cdot \frac{a}{3x^{2/3}}$. For the curve to be orthogonal this must equal -1, that is $\frac{-1}{x^2} \cdot \frac{a}{3x^{2/3}} = -1$ or equivalently, $a = 3(x^2)(x^{2/3}) = 3\left(\frac{1}{y}\right)^2\left(\frac{y}{a}\right)^2 = 3\left(\frac{1}{a^2}\right)$. In the second equality we used the equations of the curves to express x in terms of y. Comparing the two sides we see that $a = \frac{3}{a^2} \Leftrightarrow a^3 = 3 \Leftrightarrow a = \sqrt[3]{3}$.

4) Show that the equation $x^3 + e^x = 0$ has exactly one solution.

Solution. Consider the function $f(x) = x^3 + e^x$. Being an elementary function f is continuous over its domain, which in this case, is the entire number line. Moreover, we have $f(-1) = (-1)^3 + e^{-1} = \frac{1}{e} - 1 < 0$ (because $e > 1$) and $f(1) = (1)^3 + e^1 = 1 + e > 0$, therefore by the Intermediate Value Theorem there exists a number $a \in (-1, 1)$ such that $f(a) = 0$. This number is a solution of the equation $x^3 + e^x = 0$. To show that there aren't any other solutions, we argue indirectly. Assume that a number $b \neq a$ exists such that $f(b) = 0$. Since f is differentiable everywhere, by the Mean Value Theorem there is a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.$$

However the derivative function $f'(x) = 3x^2 + e^x$ only takes positive values, i.e. $f'(x) > 0$ for all x (because $x^2 \geq 0$ and $e^x > 0$ for all x). This contradicts $f'(c) = 0$ and thus the proof is concluded. We have shown that there is exactly one solution to the equation $x^3 + e^x = 0$.

Remark. Note that we did not solve the equation $x^3 + e^x = 0$, but we were still able to show that it has a (unique) solution.

5) Compute the 2nd-degree Taylor polynomial for the function $f(x) = x \ln x$ centered around the point where f attains its minimum.

Solution. First, notice that the function is only defined for positive x. We have $f'(x) = 1 + \ln x = 0 \Rightarrow x = e^{-1}$ and $f''(x) = x^{-1} \Rightarrow f''(e^{-1}) = e > 0$. Thus (by the Second Derivative Test) f has a local minimum value at $x = e^{-1}$ having the value $f(e^{-1}) = e^{-1}\ln(e^{-1}) = -e^{-1}$. Since $\lim_{x \to 0} x \ln x \overset{\text{l'H}}{=} 0$ and $\lim_{x \to \infty} x \ln x = \infty$ the local minimum at $x = e^{-1}$ is absolute.

The second-degree Taylor polynomial around $x = e^{-1}$ is

$$f(x) = f(e^{-1}) + f'(e^{-1})(x - e^{-1}) + \frac{f''(e^{-1})}{2}(x - e^{-1})^2 = -e^{-1} + \frac{e}{2}(x - e^{-1})^2 = \frac{e}{2}x^2 - x - \frac{1}{2e}.$$

6) Find a function F such that $F'(x) = x^3$ and the line $x + y = 0$ is tangent to the graph of F.

Solution. By the Power Rule for Integrals, $F'(x) = x^3$ implies that F has the general form

$$F(x) = \frac{x^4}{4} + C.$$

The line given by the equation $x + y = 0$, or equivalently $y = -x$, has slope -1. So for this line to be a tangent line the graph of F, we need F at the point of tangency $(a, F(a))$ to have the derivative $F'(a) = a^3 = -1$. This equation for a has a unique real solution, namely $a = -1$. At that point, we have $y = F(a) = F(-1) = \frac{(-1)^4}{4} + C = \frac{1}{4} + C$ (for the graph of F), but also $y = -a = -(-1) = 1$ (for the tangent line). Since the two curves meet at $(x, y) = (a, F(a))$ they must have matching y-coordinates, meaning $\frac{1}{4} + C = 1$, i.e. $C = \frac{3}{4}$. Thus the solution is

$$F(x) = \frac{x^4}{4} + \frac{3}{4} = \frac{x^4 + 3}{4}.$$